

HW #8

ALGEBRAIC NUMBER THEORY, GU4043; INSTRUCTOR: GYUJIN OH

Reading Homework. Try Exercises 4.3, 4.4, 4.5 in the textbook. Read its solutions in the back.

Question 1. Let A be an integral domain, and let $|\cdot| : A \rightarrow \mathbb{R}_{\geq 0}$ be a non-archimedean absolute value.

(1) Define

$$\begin{aligned} A^\circ &:= \{a \in A : |a| \leq 1\}, \\ A^{\circ\circ} &:= \{a \in A : |a| < 1\}. \end{aligned}$$

Show that A° is a subring of A , and $A^{\circ\circ} \subset A^\circ$ is a prime ideal.

(2) On $A[X]$, define the **Gauss norm** $|\cdot|_{\text{Gauss}} : A[X] \rightarrow \mathbb{R}_{\geq 0}$ as

$$|a_n X^n + a_{n-1} X^{n-1} + \cdots + a_0|_{\text{Gauss}} := \max(|a_n|, |a_{n-1}|, \dots, |a_0|).$$

Show that the Gauss norm is a non-archimedean absolute value on $A[X]$.

Hint. By taking $\text{Frac}(A)$, we can assume that A is a field. The only nontrivial axiom is $|f(X)g(X)|_{\text{Gauss}} = |f(X)|_{\text{Gauss}}|g(X)|_{\text{Gauss}}$ for $f(X), g(X) \in A[X]$. It is easy to show $|f(X)g(X)|_{\text{Gauss}} \leq |f(X)|_{\text{Gauss}}|g(X)|_{\text{Gauss}}$. If $|f(X)g(X)|_{\text{Gauss}} < |f(X)|_{\text{Gauss}}|g(X)|_{\text{Gauss}}$ for some $f(X), g(X) \in A[X]$, after multiplying by an appropriate element in A , one can assume, without loss of generality, that $|f(X)|_{\text{Gauss}} = |g(X)|_{\text{Gauss}} = 1$. This implies that

$$f(X), g(X) \in A^\circ[X], f(X)g(X) \in A^{\circ\circ}[X]$$

Deduce a contradiction by showing that $A^\circ[X]/A^{\circ\circ}[X] \cong (A^\circ/A^{\circ\circ})[X]$ and using (1).

(3) Use (2) to show the following¹.

Gauss's lemma. Let A be a UFD. A polynomial $f(X) \in A[X]$ is called **primitive** if the gcd of its coefficients is 1. Then, the product of primitive polynomials is primitive.

Hint. For an irreducible element $\pi \in A$, there is a normalized discrete valuation v_π on A . Taking $0 < \epsilon < 1$, one can define a non-archimedean absolute value $|\cdot|_\pi$ on A by $|x|_\pi := \epsilon^{v_\pi(x)}$.

Question 2.

(1) Let L/K be a Galois extension of p -adic local fields, and let $|\cdot|$ is a non-archimedean absolute value on L . Show that any element $\sigma \in \text{Gal}(L/K)$ is an isometry. In other words, show that $|x - y| = |\sigma(x) - \sigma(y)|$ for any $x, y \in L$.

Hint. Use Theorem 4.44.

¹A special version was stated in Theorem 1.11.

- (2) **(Krasner's lemma)** Let K be a p -adic local field. Let $f(X) \in K[X]$ be an irreducible separable monic polynomial, and let L/K be a separable extension of local fields containing the splitting field of $f(X)$; in other words, $f(X)$ factorizes in $L[X]$ as

$$f(X) = \prod_{i=1}^n (X - \alpha_i), \quad \alpha_i \in L.$$

Let $|\cdot|$ be a non-archimedean absolute value on L . Let $\beta \in L$ be such that

$$|\beta - \alpha_1| < |\alpha_j - \alpha_1| \quad \text{for all } j \neq 1.$$

Show that $K(\alpha_1) \subset K(\beta)$.

Hint. Enlarge L so that L/K is Galois. If $K(\alpha_1) \not\subset K(\beta)$, there should be $\sigma \in \text{Gal}(L/K)$ such that $\sigma(\alpha_1) \neq \alpha_1$ and $\sigma(\beta) = \beta$. Use (1) to deduce a contradiction.

- (3) Let K be a p -adic local field, and $|\cdot|$ be a non-archimedean absolute value on K (or its unique extension on any finite extension of K). Let L/K be a finite Galois extension. Show that there exists $0 < c_{L/K} \leq 1$ (depending only on $L, K, |\cdot|$) such that the following holds.

Lemma (Fontaine). Let E/K be any Galois extension, and let $0 < \epsilon < c_{L/K}$. Define

$$\mathfrak{m}_E^\epsilon = \{x \in \mathcal{O}_E : |x| \leq \epsilon\}.$$

If there exists an \mathcal{O}_K -algebra homomorphism $\mathcal{O}_L \rightarrow \mathcal{O}_E/\mathfrak{m}_E^\epsilon$, there exists an \mathcal{O}_K -algebra homomorphism $\mathcal{O}_L \rightarrow \mathcal{O}_E$.

Hint. By Theorem 4.38(1), $\mathcal{O}_L = \mathcal{O}_K[\alpha]$ for some $\alpha \in \mathcal{O}_L$. Let $f(X) \in \mathcal{O}_K[X]$ be the minimal polynomial of α over \mathcal{O}_K . What the problem is asking is, if there is $\beta \in \mathcal{O}_E$ such that $|f(\beta)| \leq \epsilon$, whether there is $\beta' \in \mathcal{O}_E$ such that $f(\beta') = 0$. Use (2).