

HW #12 (OPTIONAL)

ALGEBRAIC NUMBER THEORY, GU4043; INSTRUCTOR: GYUJIN OH

Reading Homework. Try Exercises 6.1, 6.2, 6.5, 6.8 in the textbook. Read the solutions in the back.

Question 1. In this question, we will prove the quartic reciprocity law in an analytic way.

Let $K = \mathbb{Q}(\sqrt{-1})$, embedded as a subfield of $\mathbb{C}$ as $K \hookrightarrow \mathbb{C}$, $\sqrt{-1} \mapsto i$, and we will identify $\sqrt{-1} \in K$ with $i \in \mathbb{C}$. Let $\pi = 1 - i$. Recall that in HW#11, Question 3, we defined that, for a rational prime $p \equiv 1 \pmod{4}$, a **primary number** is a generator of a prime ideal dividing (p) that is congruent to 1 $\pmod{\pi^3}$. For each prime ideal of $\mathcal{O}_K$ dividing p , there is a unique generator that is a primary number.

Let π_1, π_2 be primary numbers of K dividing rational primes $p_1, p_2 \equiv 1 \pmod{4}$, and suppose that $p_1 \neq p_2$. For $j = 1, 2$, we let $\chi_j(n) := \left(\frac{n}{\pi_j}\right)$ as a Dirichlet character mod p_j .

- (1) For $j = 1, 2$, show that $G(\chi_j)^4 = p_j J(\chi_j, \chi_j)^2$.

Hint. Show that χ_j^2 , when evaluated at integers, is the Legendre symbol mod p_j .

- (2) For $j = 1, 2$, show that

$$J(\chi_j, \chi_j) = \begin{cases} \pi_j & \text{if } p_j \equiv 5 \pmod{8}, \\ -\pi_j & \text{if } p_j \equiv 1 \pmod{8}. \end{cases}$$

Hint. Let $p_j = 2k_j - 1$. Show that $J(\chi_j, \chi_j) \equiv \chi_j(k_j)^2 + 2(k_j - 2) \pmod{\pi^3}$ by using that any $u \in \mathcal{O}_K$ satisfies $u \equiv 1 \pmod{\pi}$. Show that

$$\chi_j(k_j)^2 = \chi_j(2)^2 = \begin{cases} 1 & \text{if } p_j \equiv 1 \pmod{8}, \\ -1 & \text{if } p_j \equiv 5 \pmod{8}. \end{cases}$$

Use Lemma 6.25 to finish the proof.

- (3) Note that, for $j = 1, 2$, $\bar{\pi}_j$, the complex conjugate of π_j , is a primary number. Show that

$$\chi_1(p_2) = \chi_2(p_1 \bar{\pi}_1^2).$$

Hint. Mimic the analytic proof of cubic reciprocity law as in the textbook, by computing $G(\psi_1)^{p_2}$ in two different ways, where $\psi_j(n) := \left(\frac{n}{\pi_j}\right)$.

- (4) Show that

$$\left(\frac{p_2}{\pi_1}\right) = \left(\frac{\pi_1}{p_2}\right) \quad \left(\text{namely, } \left(\frac{\pi_2}{\pi_1}\right) \left(\frac{\bar{\pi}_2}{\pi_1}\right) = \left(\frac{\pi_1}{\pi_2}\right) \left(\frac{\pi_1}{\bar{\pi}_2}\right)\right).$$

Hint. Use that $\overline{\chi_2(\bar{\pi}_1)} = \psi_2(\pi_1)$.

(5) Let $q \equiv 3 \pmod{4}$ be a rational prime. Show that

$$\left(\frac{-q}{\pi_1}\right) = \left(\frac{\pi_1}{-q}\right).$$

Hint. Compute $G(\chi_1)^{q+1}$ in two different ways. Along the way, it is useful to note that $\bar{\pi}_1 \equiv \pi_1^q \pmod{q}$.

(6) Combine (4) and (5) to show that, for any integer n with $n \equiv 1 \pmod{4}$ and $(n, p_1) = 1$,

$$\left(\frac{n}{\pi_1}\right) = \left(\frac{\pi_1}{n}\right).$$

(7) For any odd integers $a, b \in \mathbb{Z}$ with $(a, b) = 1$, show that $\left(\frac{a}{b}\right) = 1$.

(8) For $j = 1, 2$, let

$$\tilde{\pi}_j := \begin{cases} \pi_j & \text{if } \pi_j \equiv 1 \pmod{4}, \\ -\pi_j & \text{if } \pi_j \equiv 3 \pmod{4}. \end{cases}$$

In particular, $\tilde{\pi}_j = a_j + b_j i$ with $a_j \equiv 1 \pmod{4}$. Show that

$$\left(\frac{\pi_1}{\pi_2}\right) \left(\frac{\pi_2}{\pi_1}\right)^{-1} = \left(\frac{\tilde{\pi}_1}{\pi_2}\right) \left(\frac{\tilde{\pi}_2}{\pi_1}\right)^{-1}.$$

(9) Show that

$$\left(\frac{\tilde{\pi}_1}{\pi_2}\right) \left(\frac{\tilde{\pi}_2}{\pi_1}\right)^{-1} = \left(\frac{a_2}{\pi_2}\right)^{-1} \left(\frac{a_1}{\pi_1}\right) \left(\frac{a_1 a_2 + b_1 b_2}{\bar{\pi}_1 \pi_2}\right).$$

Use (6) and (7) to show that

$$\left(\frac{\tilde{\pi}_1}{\pi_2}\right) \left(\frac{\tilde{\pi}_2}{\pi_1}\right)^{-1} = \left(\frac{i}{a_2}\right)^{-1} \left(\frac{i}{a_1}\right) \left(\frac{i}{a_1 a_2 + b_1 b_2}\right).$$

(10) If $a \in \mathbb{Z}$ is congruent to 1 $\pmod{4}$, show that

$$\left(\frac{i}{a}\right) = (-1)^{\frac{a-1}{4}}.$$

(11) Deduce the quartic reciprocity law,

$$\left(\frac{\pi_1}{\pi_2}\right) \left(\frac{\pi_2}{\pi_1}\right)^{-1} = (-1)^{\frac{N_{K/\mathbb{Q}}(\pi_1)-1}{4} \frac{N_{K/\mathbb{Q}}(\pi_2)-1}{4}}.$$

Question 2. Use various methods we have learned in class to show that $h_{\mathbb{Q}(\sqrt{-31})} = 3$ in three different ways.

Question 3. Let $K/\mathbb{Q}$ be a cubic Galois field. Note that this forces K to only have real embeddings, and therefore $\text{rank } \mathcal{O}_K^\times = 2$. Note that $\text{disc}(K)$ is a perfect square by Question 1 of Midterm.

- (1) Let $\text{disc}(K) = f^2$ for $f > 0$. Use the Conductor-discriminant formula, Theorem 7.5, to show that $K \subset \mathbb{Q}(\zeta_f)$.

Hint. Note that, if χ is a nontrivial character of $\text{Gal}(K/\mathbb{Q})$, then the characters of $\text{Gal}(K/\mathbb{Q})$ are $\{1, \chi, \chi^{-1}\}$. Thus, if m is the minimal integer such that $K \subset \mathbb{Q}(\zeta_m)$ (which exists by the Kronecker–Weber theorem), show that the Conductor-discriminant formula implies that $\text{disc}(K) = m^2$.

- (2) Let χ be a nontrivial Dirichlet character of conductor f and order 3 (i.e., $\chi^3 = 1, \chi \neq 1$). Show that

$$(**) \quad h_K R_K = \left| \sum_{a=1}^{\lfloor \frac{f}{2} \rfloor} \chi(a) \log \left(\sin \left(\frac{\pi a}{f} \right) \right) \right|^2.$$

Hint. Use analytic class number formula.

- (3) Now consider $K = \mathbb{Q}(\zeta_7)^+$ (cf. Exercise 2.5). Recall that $\frac{\zeta_7^a - 1}{\zeta_7 - 1}$, $1 \leq a \leq 6$, is a unit in $\mathbb{Q}(\zeta_7)$ (**cyclotomic units**). Show that

$$\epsilon_1 := \zeta_7^3 \frac{\zeta_7^2 - 1}{\zeta_7 - 1}, \quad \epsilon_2 := \zeta_7^{-1} \frac{\zeta_7^3 - 1}{\zeta_7 - 1},$$

are units in K .

- (4) Let $\sigma_1 : K \hookrightarrow \mathbb{R}$ be defined as $\zeta_7 + \zeta_7^{-1} \mapsto 2 \cos(2\pi/7)$, and $\sigma_2 : K \hookrightarrow \mathbb{R}$ be defined as $\zeta_7 + \zeta_7^{-1} \mapsto 2 \cos(6\pi/7)$. Show that

$$\left| \det \begin{pmatrix} \log(|\sigma_1(\epsilon_1)|) & \log(|\sigma_1(\epsilon_2)|) \\ \log(|\sigma_2(\epsilon_1)|) & \log(|\sigma_2(\epsilon_2)|) \end{pmatrix} \right| \sim 0.52545 \dots$$

Deduce that $\langle \epsilon_1, \epsilon_2 \rangle$ generate a finite index subgroup of $\mathcal{O}_K^\times$.

- (5) Compute the right hand side of (**) when $K = \mathbb{Q}(\zeta_7)^+$, and show that

$$\text{RHS of } (**), \text{ when } K = \mathbb{Q}(\zeta_7)^+ \sim 0.52545 \dots$$

- (6) Show that $h_K = 1$.

- (7) Let C^+ be the group (multiplicatively) generated by $\pm 1, \epsilon_1, \epsilon_2$. Show that¹ $\mathcal{O}_K^\times = C^+$.

¹cf. Theorem 7.16