HW #10

ALGEBRAIC NUMBER THEORY, GU4043; INSTRUCTOR: GYUJIN OH

Reading Homework. Try Exercise 5.1 in the textbook. Read its solutions in the back.

Question 1. In Exercise 4.6, we have determined all 7 quadratic extensions of $\mathbb{Q}_2$. For each quadratic extension $K/\mathbb{Q}_2$, compute the local conductor $\mathfrak{f}_{K/\mathbb{Q}_2}$.

Hint. Note that $\mathfrak{f}_{K/\mathbb{Q}_2}$ can never be equal to 1 in this case; do you see why?

Question 2. Use local class field theory to show the following.

(1) Let L/K be a totally ramified abelian extension of local fields. Suppose that L/K is tamely ramified. Let k_K be the residue field of K. Show that $[L:K] \leq \#k_K - 1$. If $[L:K] = \#k_K - 1$, we say that L is a **maximal totally tamely ramified abelian extension** of K.

(2) Show that there are exactly $\#k_K - 1$ many distinct maximal totally tamely ramified abelian extensions of K.[1]

(3) We know that $\mathbb{Q}_3(\zeta_3)$ is a maximal totally tamely ramified abelian extension of $\mathbb{Q}_3$. Find another maximal totally ramified abelian extension of $\mathbb{Q}_3$ that is not isomorphic to $\mathbb{Q}_3(\zeta_3)$.

Hint. You may find Exercise 4.6 helpful.

Question 3. Let $K/\mathbb{Q}_p$ be a finite extension. For a finite extension L/K of degree n, show that

$$\mathfrak{f}_{L/K} \leq e_{K/\mathbb{Q}_p}\left(v_p(n) + \frac{1}{p-1}\right) + 1.$$

Hint. Let π be a uniformizer of K and $N = \left\lfloor e_{K/\mathbb{Q}_p}\left(v_p(n) + \frac{1}{p-1}\right) + 1 \right\rfloor$. Use Exercise 4.4 to deduce that every element in $1 + \pi^N \mathcal{O}_K$ is an n-th power. Then, use local class field theory.

[1] You might be confused why there are more than one maximal such extensions, unlike the case of a maximal tamely ramified extension (which is unique). The point is that, even though the tamely-ramified-ness is preserved under compositum, the **totally-ramified-ness** is not preserved under compositum.